The Ewald–Oseen extinction theorem and extinction lengths

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We give an elementary demonstration of the extinction theorem for electromagnetic waves at normal incidence on a plane surface of a medium. We stress that the extinction of the incident wave and its replacement with a wave of index of refraction n takes place throughout the medium rather than in the surface layers. Although the extinction theorem is usually thought to apply only to dielectrics, we extend the theorem to include conductors. We use the macroscopic fields in which the contributions of the oscillating dipoles in a dielectric or the induced currents in a conductor are already summed. Our elementary derivation of the extinction theorem should be readily accessible to advanced undergraduates since it depends only on the superposition principle and the solution of the wave equation. An analogy is made between this extinction and the cancellation of the electric field inside a conductor placed in a static electric field. We also study the more advanced case of propagation of radiation in a dilute random medium in which the wavelength is small relative to the interparticle distance and find that an analogous extinction of the incident wave takes place. Furthermore, for the dilute random medium, we estimate the length into the medium for which a large fraction, $(1-1/e)$, of the incident radiation has interacted with the particles making up the medium. This length is much larger than any lengths associated with the extinction theorem itself.

I. INTRODUCTION

It is well known that the speed of light in a medium is given by c/n where n is the index of refraction of the medium. This is usually shown by solving the wave equation in which the macroscopic electric and magnetic fields are already the superposition of the incident wave and the radiation of the oscillating atoms that make up the medium. The interference of these two components is not manifest but was demonstrated by Ewald and Oseen in their famous extinction theorem.[1] They showed that, inside a dielectric, the radiation from the atoms exactly cancels the electromagnetic field of the incident wave and replaces it by a field corresponding to a wave with speed c/n.

Unfortunately, their calculation is not readily accessible to undergraduates because of the advanced mathematics (a typical derivation involves three-dimensional Green's functions, electromagnetic radiation theory, and the resolution of an integral equation). Over the years many papers[2–8] have appeared in this journal on the extinction theorem, transmission and reflection, and the index of refraction. Perhaps the most ambitious is the paper by Fearn *et al.*[2] who show that, in a dielectric continuous on the scale of a wavelength, the extinction of the incident wave and its replacement with a wave of index of refraction n takes place throughout the medium rather than in the surface layers. They accomplished this by showing that the near, intermediate, and far fields of the oscillating dipoles superimpose to give waves traveling in the forward and backward directions only. This is very satisfying. On the other hand, the extinction theorem is about macroscopic fields and it is not necessary to explicitly sum the complicated microscopic fields. In Sec. II, we show in a remarkably simple way how the extinction follows directly from the wave equation for the macroscopic fields and the superposition principle. That the fields of the oscillating dipoles give such traveling waves is implicit in our work, that is, the contributions of all the dipoles are automatically summed in the macroscopic fields. The idea of our approach is to separate the various sources that produce the electromagnetic field, solve Maxwell equations separately for the field of each source, and, finally, superimpose the resulting fields.

To avoid unnecessary complications, we consider the simple situation of an electromagnetic wave of pure frequency at normal incidence on a medium filling half the space (the region $z \geqslant 0$). The electric field at a point in space is the sum of the electric fields due to all the various sources. In our situation, we can separate the sources into two groups: first the sources external to the medium that produce the incident wave and second the radiating atoms making up the medium. We denote the contribution to the electric field due to the external sources by $\mathbf{E}_{\text{vac}}(z,t)$. This is, of course, the electric field of the incident wave traveling to the right in the positive z direction. This contribution to the electric field is present everywhere in space, outside the medium as well as deep inside. The electric field produced by the radiation of all the atoms in the medium constitutes the other contribution and we refer to it as the "radiation field" $\mathbf{E}_{\text{rad}}(z,t)$. The resulting electric field at z is just the superposition[9]

$$\mathbf{E}(z,t)=\mathbf{E}_{\text{vac}}(z,t)+\mathbf{E}_{\text{rad}}(z,t). \tag{1}$$

With this notation, the extinction theorem states that the radiation field in the medium is equal to

$$\mathbf{E}_{\text{rad}}(z,t)=-\mathbf{E}_{\text{vac}}(z,t)+\mathbf{E}_{T}(z,t), \tag{2}$$

that is, the radiation field exactly cancels the incident wave and creates a new or transmitted wave, $\mathbf{E}_T(z,t)$, traveling at speed c/n within the medium. Outside the medium $(z<0)$, the radiation field takes the form of a wave traveling at speed

c in the direction opposite to the incident wave and corresponds to the reflected wave.

The same principle of superimposing the fields due to the various sources can be used, for example, in the description of the electric field inside and outside a conductor placed in a uniform static electric field. Assuming that the charges induced on the surface of the conductor do not disturb the distant charges producing the uniform field, the electric field at a given point in space is the sum of the uniform electric field plus the field produced by the surface charge density induced on the sphere. Outside the conductor, the resulting electric field is nonuniform, but, inside the conductor, the two contributions to the field sum to zero. Interestingly, we calculate the electric fields due to the induced charges as if there were no medium, that is, the governing equations are Maxwell's equations in vacuum. Similarly, the uniform electric field is the same with or without the conductor under the assumption that the charges producing the uniform electric field are not disturbed by the introduction of the conductor. This follows directly from the linear relationship between the electric field and its sources as expressed in the differential form of Gauss' law,

$$\epsilon_0 \nabla \cdot \mathbf{E}(r) = \rho(r). \tag{3}$$

Notice the broad meaning[10] we give to the word sources. Here we take the fields **E** and **B** to be fundamental and the source terms to include not only the true charges and currents but also induced quantities. These may include the fields themselves and their time derivatives such as polarization charges and currents. Some[11] would restrict the term ''source'' to the primary cause or origin, presumably those charges and currents over which one has direct control. Maxwell's equations supplemented by the constitutive relations can be solved with just this information alone, but such a limitation is too constraining for our purposes and for many applications.

For the conductor placed in a uniform static electric field we can therefore say that the induced charges produce an electric field that exactly cancels the original uniform field everywhere within the conductor. This ''extinction'' of the field within the conductor is an electrostatic analog of the extinction of the vacuum wave by the radiation field.

There can be some confusion about the word extinction as it is used in the literature. For example, Jackson[12] argues that the incident wave cannot be extinguished immediately when entering the medium, but only after an ''extinction distance.'' This may suggest the incorrect idea that the extinction theorem refers to the incident wave being absorbed and re-emitted by layers of the medium nearest the surface with the re-emitted wave as the new wave traveling at the speed c/n. What we mean by extinction is that the incident wave produced by the external sources is canceled everywhere within the medium by the induced radiation fields, just as the uniform electric field is canceled everywhere within the conductor by the induced charges. In the usual macroscopic model in which matter is taken to be continuous, the extinction is perfect even at the surface. More on this later.

The extinction theorem is usually thought to apply only to dielectrics, but the physics is the same for radiation incident on other media, such as a conductor or a plasma, where again the radiation fields interfere with the incident wave. We will see that in a conductor, the radiation fields cancel the vacuum wave, leaving a transmitted wave with an amplitude decaying exponentially with distance. To include these media in our derivation, we will speak of the radiation produced by induced currents in the medium. In a dielectric these currents would be polarization currents (collective oscillations of the atoms) and in a conductor they would be true currents. In the next section we follow the procedure described above to demonstrate the extinction theorem for a continuous medium.

In Sec. III we write the solution of the wave equation as an integral equation in which the response of the medium is given by a single term. This integral equation can be easily compared to earlier work.

In Sec. IV we treat a dilute random medium by looking at the scattering of the individual particles that make up the medium. Here the wavelength is less than the interparticle distances and the radiation fields are incoherent except in the forward scattering direction. Although the extinction theorem is usually thought to apply only to a continuous dielectric medium with well-defined boundaries, we find that an analogous extinction also takes place in this case of a dilute random medium.

In Sec. V, scattering in a dilute random medium is used to discuss three meanings of extinction length.

II. ELEMENTARY DEMONSTRATION OF THE EXTINCTION THEOREM FOR NORMAL INCIDENCE

Consider a plane monochromatic electromagnetic wave impinging normally on a uniform isotropic medium filling the half-space $z \geqslant 0$. Let the z axis point in the direction of propagation and assume that the wavelength is much larger than the average separation of atoms so that the medium can be considered continuous. We use the usual macroscopic E and B fields and take the medium to be nonmagnetic and neutral so that Maxwell's equations read

$$\nabla \cdot \mathbf{E} = 0 \tag{4a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4b}$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \tag{4c}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \partial \mathbf{E}/\partial t. \tag{4d}$$

Here **J** includes the true and polarization currents induced in the medium but no magnetization current. We assume a linear relationship between the current and the electric field as expressed by Ohm's law with conductivity σ,

$$\mathbf{J} = \sigma \mathbf{E}. \tag{5}$$

This current density will be a source of the radiation fields. Since the current density **J** can be out of phase with respect to **E**, we allow **E**, **J**, and σ to be complex in the usual way, with the understanding that only the real part of the fields has physical significance.

Since Eqs. (4) are linear in the fields, their solution can be written as the superposition,

$$\mathbf{E} = \mathbf{E}_{\text{vac}} + \mathbf{E}_{\text{rad}}, \quad \mathbf{B} = \mathbf{B}_{\text{vac}} + \mathbf{B}_{\text{rad}}, \tag{6}$$

where

$$\nabla \cdot \mathbf{E}_{\text{rad}} = 0, \tag{7a}$$

$$\nabla \cdot \mathbf{B}_{\text{rad}} = 0, \tag{7b}$$

$$\nabla \times \mathbf{E}_{\text{rad}} = -\partial \mathbf{B}_{\text{rad}}/\partial t, \tag{7c}$$

$$\nabla \times \mathbf{B}_{\text{rad}} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \partial \mathbf{E}_{\text{rad}}/\partial t, \tag{7d}$$

and a second set of Maxwell's equations with no current density for the vacuum fields. Note that these two sets of Maxwell's equations are coupled since the vacuum field appears in the current density. It should be noted that the decomposition of the **E** and **B** fields into a sum of ''contributions'' is, to a certain extent, arbitrary since any source can be added to one set of equations and subtracted from the other. We chose our decomposition so that the fields labeled ''vac'' satisfy the wave equation in vacuum and can therefore be identified with the fields of the incident wave for $z<0$. For a monochromatic wave at normal incidence, the vacuum wave takes the form

$$\mathbf{E}_{\rm vac}(z,t)=\mathbf{E}_{\rm vac}\exp[i(kz-\omega t)], \tag{8}$$

for all z with $k=\omega/c$.

The remaining contribution to the electric field, $\mathbf{E}_{\rm rad}$, has its origin entirely in the medium. The source associated with these fields is the current density **J** which represents the oscillating charges of the atoms or the moving electrons in a conductor. Applying the curl to Eq. (7c) and substituting Eq. (7d) gives

$$\nabla\times\nabla\times\mathbf{E}_{\rm rad}=-\partial(\mu_0\mathbf{J}+\mu_0\epsilon_0\partial\mathbf{E}_{\rm rad}/\partial t)/\partial t, \tag{9}$$

where, for fields restricted to Cartesian components, the triple cross product reduces to

$$\nabla\times\nabla\times\mathbf{E}_{\rm rad}=-\nabla^2\mathbf{E}_{\rm rad}+\nabla(\nabla\cdot\mathbf{E}_{\rm rad}). \tag{10}$$

Since all the fields have the same time dependence, $\exp[-i\omega t]$, the time derivatives in Eq. (9) are easily done. Then from Eqs. (7a) and (10), we see that the radiation field satisfies the following inhomogeneous wave equation,

$$\nabla^2\mathbf{E}_{\rm rad}+\mu_0\omega^2(\epsilon_0+i\sigma/\omega)\mathbf{E}_{\rm rad}=-i\mu_0\omega\sigma\mathbf{E}_{\rm vac}(z), \tag{11}$$

where the right-hand side is given. From inspection, a particular solution to this equation is

$$\mathbf{E}_{\rm rad}^{P}=-\mathbf{E}_{\rm vac}(z). \tag{12}$$

For the complete solution we need to add to this particular solution the general solution of the homogeneous equation, that is, a superposition of plane waves traveling in arbitrary directions,[13]

$$(\mathbf{E}_{\rm rad}^{c})_i=\int g_i(\theta,\phi)\exp(i\mathbf{k}'\cdot\mathbf{r})\,d\Omega, \tag{13}$$

where θ and ϕ are the polar and azimuthal angles of $\mathbf{k}'$ and $d\Omega$ is the element of solid angle. The subscript i stands for the Cartesian components. The magnitude of $\mathbf{k}'$ is given by

$$k'^2=\mu_0\epsilon_0\omega^2\left(1+i\frac{\sigma}{\epsilon_0\omega}\right). \tag{14}$$

Note that we have taken the solution as a coherent superposition of plane waves. This will certainly be true for the cases where the medium can be treated as continuous since the radiation fields from the various elements of the medium will arrive at a given point with a definite phase relationship which does not change with time. The same cannot be said for a gaseous medium because of fluctuations. More on this in Sec. IV.

Because of symmetry, the fields should be the same at all points in a plane perpendicular to the z axis. Hence,

$$\mathbf{k}'\cdot\mathbf{a}=0, \tag{15}$$

where **a** is a displacement in the x-y plane perpendicular to the z axis. Furthermore, since there are no boundaries to the right, we expect waves traveling only to the right. The solution to the homogeneous equation will be the transmitted wave,

$$\mathbf{E}_{\rm rad}^{c}=\mathbf{E}_T\exp(ik'z). \tag{16}$$

Adding this to the particular solution given in Eq. (12) gives

$$\mathbf{E}_{\rm rad}=-\mathbf{E}_{\rm vac}(z)+\mathbf{E}_T\exp(ik'z) \tag{17}$$

for the solution of the radiation fields. Adding this to the vacuum field gives the final result for the field within the medium,

$$\mathbf{E}(z)=\mathbf{E}_T\exp(ik'z). \tag{18}$$

This remarkably simple demonstration shows that the radiation field contribution to the electric field has the effect of canceling the incident wave $\mathbf{E}_{\rm vac}\exp(ikz)$ and of creating a new wave $\mathbf{E}_T\exp(ik'z)$ traveling at speed c/n, where

$$n=ck'/\omega=\sqrt{1+i\frac{\sigma}{\epsilon_0\omega}}. \tag{19}$$

This extinction and replacement with a new wave is often described as the interference between the radiation field given by Eq. (17) and the incident wave.

To find the usual formula for the index of refraction of a linear isotropic dielectric, we need to write the conductivity of the dielectric as a function of its susceptibility χ_e. In such a dielectric, an electric field **E** will induce a dipole moment per unit volume proportional to the electric field, $\mathbf{P}=\epsilon_0\chi_e\mathbf{E}$. When the electric field changes, the induced charges move and produce a volume current density (polarization current) given by $\partial\mathbf{P}/\partial t$. Since the electric field of the wave has the time dependence $\exp[-i\omega t]$, we obtain

$$\mathbf{J}=-i\epsilon_0\omega\chi_e\mathbf{E}, \tag{20}$$

which gives the conductivity,

$$\sigma=-i\epsilon_0\omega\chi_e. \tag{21}$$

Finally, substitution of this conductivity into Eq. (19) gives

$$n=\sqrt{1+\chi_e}, \tag{22}$$

the index of refraction in terms of the susceptibility. In a conductor, σ is real and positive so that the index of refraction has a nonzero imaginary part; the transmitted wave decays exponentially with distance as can be seen from Eqs. (18) and (19).

For $z<0$ where $\sigma=0$ the solution of Eq. (11) is easily seen to be

$$\mathbf{E}_{\rm rad}(z)=\mathbf{E}_R\exp(-ikz), \tag{23}$$

where we have imposed the condition of a traveling wave to the left. This will be the reflected wave. The amplitudes of the reflected and the transmitted waves can be related to the incident amplitude by applying the usual boundary conditions at the interface[14] to obtain the Fresnel relations for normal incidence.

In conclusion, we have shown explicitly, for the case of normal incidence on a homogeneous and isotropic medium, how the extinction of the vacuum wave follows directly from the solution of the wave equations for the macroscopic fields, with the contribution of the medium treated separately. The result is independent of the boundary conditions at $z=0$ and also independent of the direction of propagation of the resulting waves within the medium. Notice that if the dielectric medium is of finite thickness, waves within the medium will travel in the negative as well as the positive z rection.

III. THE INTEGRAL SOLUTION

In order to compare our elementary presentation to the traditional approach, we recast the wave equation,

$$\frac{d^2}{dz^2}\mathbf{E}(z)+\mu_0\epsilon_0\omega^2\mathbf{E}(z)=-i\omega\mu_0\sigma(z)\mathbf{E}(z), \tag{24}$$

into an integral equation. Here the electric field is the total electric field as given in Eq. (1). Setting $\sigma(z)=0$ gives what we will call the homogeneous equation. Notice that for $z\geqslant 0$ this equation describes the transmitted wave within the medium. In this region $\sigma(z)$ is the constant σ and for $z<0$ it is zero. Next we write the solution to this wave equation for all z as an integral equation by using a Green's function defined by

$$\left(\frac{d^2}{dz^2}+k^2\right)G(z,z')=\delta(z-z'). \tag{25}$$

This Green's function can be constructed in terms of two independent solutions to the homogeneous equation,[15] for example, $\exp(ikz)$ and $\exp(-ikz)$. We take

$$G(z,z')=\begin{cases}C\exp[ik(z-z')], & z>z',\\ C\exp[ik(z'-z)], & z<z',\end{cases} \tag{26}$$

which is continuous at $z=z'$. Here we have imposed the boundary condition that $G(z,z')$ be an outgoing wave for z large, the same condition as we demand for the solution. C is a constant determined so that the first z derivative of the Green's function has a unit step at $z=z'$. We find that

$$C=\frac{1}{2ik}. \tag{27}$$

It is easy to see that a particular solution to the wave equation is

$$\mathbf{E}^P(z)=-i\omega\mu_0\sigma\int_0^\infty G(z,z')\mathbf{E}(z')\,dz', \tag{28}$$

by applying the operator

$$\frac{d^2}{dz^2}+k^2 \tag{29}$$

to both sides. On the right the operator goes under the integral sign and operates on the Green's function to give a δ function or, alternatively, the integral can be done by parts. Notice that this particular solution can be interpreted as the contribution of the medium since it is proportional to σ. For the complete solution we must add the solution of the homogeneous equation such that for large positive z we have only outgoing waves while for large negative z we have incoming and outgoing waves. The solution is

$$\mathbf{E}(z)=\mathbf{E}_{\text{vac}}\exp(ikz)-\omega\mu_0\frac{\sigma}{2k}\int_0^\infty \exp(ik|z-z'|)\mathbf{E}(z')\,dz'. \tag{30}$$

For $z\geqslant 0$ this gives the transmitted wave $\mathbf{E}_T$ while for $z<0$ this gives the incident wave $\mathbf{E}_{\text{vac}}$ plus the reflected wave.

For a dielectric, the integral in Eq. (30) is interpreted as the contribution to the radiation fields by the induced oscillations of the dipoles making up the medium. This is exactly the same result as obtained by Fearn *et al.*[2] who, as mentioned earlier, summed the complicated dipole fields. Obviously one can use Maxwell's equations to find the fields of an oscillating dipole first and then sum over the medium or solve Maxwell's equations directly for the continuous source of dipoles that make up the medium. This latter procedure automatically sums the dipole fields. The reader is referred to the work of Fearn *et al.*[2] for the details of the solution of the integral equation. This solution demonstrates the extinction theorem.

IV. A DILUTE RANDOM MEDIUM

It is usually thought that the extinction theorem is restricted to continuous media where the wavelength is much larger than the interparticle spacing. In this section we find that an analogous extinction takes place in a dilute medium where the wave length is much smaller than the interparticle distances. We investigate the scattering of light traveling through a medium of many randomly distributed particles such as a dilute gas. We start with a truly microscopic approach since we will consider the scattering of each of the particles that make up the medium. But then we sum over a large volume to get the macroscopic fields. We follow the analysis of Roger Newton in his book[16] on scattering theory where many of the details can be found.

The waves scattered by the particles in directions other than forward are incoherent since they come from a random distribution of scattering centers which changes with time. Thus the intensities add rather than the amplitudes so that energy is scattered out of the beam. Compare this with the continuous medium studied in Sec. II where waves scattered in directions other than forward interfere destructively so that all the scattered radiation goes in the forward direction.

It should be noted in passing that energy can be scattered out of the beam coherently if the scattering centers are in an orderly arrangement, for example, as in x-ray diffraction. We do not consider such cases here.

As in the case of the continuous medium, we will consider the random medium to fill the half-space $z\geqslant 0$ and to be isotropic and homogeneous on some scale. The incident plane wave travels in the positive z direction. Furthermore, we will consider scattering only in the forward direction where the scattering is coherent. The coherence follows from consideration of the waves arriving at a field point (x_0,y_0,z_0) from a particle with the same coordinates x_0 and y_0 but with $z<z_0$. Careful consideration shows that changing the z coordinate of the scatterer does not change the phase of the waves arriving at the field point; that is, waves arriving at the field point from various scatterers at different z's but approximately the same x_0 and y_0 will be in phase. For the scattering, we take the asymptotic form of the fields to be an incoming polarized plane wave and a spherical outgoing wave,

$$\mathbf{E}(z)\approx\mathbf{E}_0\exp(ikz)+\mathbf{E}_{\text{scat}}\exp(ikr)/r. \tag{31}$$

For scattering in the forward direction, we assume the polarization to be unchanged. Then the scattered wave is proportional to the incident wave,

$$\mathbf{E}_{\text{scat}}=f(0)\mathbf{E}_0, \tag{32}$$

where $f(0)$ is the forward scattering amplitude.

Following Newton, we take the interparticle distance, D, to be very large with respect to the particle size, R, so that multiple scattering can be neglected. Furthermore, the interparticle distance is taken to be much larger than the wavelength of the radiation. This last condition assures us that we only need to consider the *radiation* fields of the oscillating dipoles that make up the medium. Newton takes a slab of material of thickness dz which is infinitesimal in that the beam does not change appreciably in traversing dz yet large enough to contain many particles, that is, dz is much larger than the interparticle distance but much smaller than the length of a column of particles for which the sum of the cross-sectional areas of the particles equals the cross-sectional area of the column. Now consider a field point that is a distance d from the slab. This distance is taken small compared to the thickness of the slab but still large compared to the interparticle distance, that is,

$$D \ll d \ll dz. \tag{33}$$

As described by Newton, the medium looks continuous and the field point lies on the surface of the infinitesimal slab dz under a microscope of ''intermediate'' power.

Next we must sum the contributions at the field point from the particles that make up the slab. Only those particles within a narrow cone perpendicular to the slab with apex at the field point a distance d from the slab contribute to the electric field at that point. The result is

$$\mathbf{E}(z) = \mathbf{E}_0 \exp(ikz) + 2\pi i N k^{-1} dz f(0) \mathbf{E}_0 \exp(ikz), \tag{34}$$

where N is the density of particles. For the details of this calculation we refer the reader to Newton's book.[16]

The scattering described by Eq. (34) takes place throughout the medium so that the amplitude factor of the incident wave, $\mathbf{E}_0$, will depends on z. To account for this, we write the amplitude as $\mathbf{A}(z)$ and take $\mathbf{A}(0) = \mathbf{E}_0$, the amplitude of the wave incident at $z=0$. In the thickness of the slab, the amplitude has changed by

$$d\mathbf{A}(z) = 2\pi i N k^{-1} dz f(0) \mathbf{A}(z). \tag{35}$$

If the forward scattering amplitude $f(0)$ were real, then the change in the wave is purely imaginary. In that case the intensity of the wave is not attenuated as it travels through the medium. Integrating Eq. (35) gives

$$\mathbf{A}(z) = \mathbf{E}_0 \exp[2\pi i N k^{-1} f(0) z], \tag{36}$$

so that

$$\mathbf{E}(z) = \mathbf{E}_0 \exp(inkz), \tag{37}$$

where the index of refraction is

$$n = 1 + 2\pi N f(0)/k^2. \tag{38}$$

These equations clearly show that it is the interference of the scattered wave (radiation fields) and the incident wave that produces the refracted wave.

We can calculate the index of refraction for the dilute gas by using the simple model of an atom as an ion with an electron attached by a linear restoring force of strength $m(\omega_0)^2$. The scattering amplitude can easily be obtained from the radiation fields of an oscillating dipole,

$$\mathbf{E}_{\mathrm{scat}} = [(\mathbf{k} \times \mathbf{p}_0) \times \mathbf{k}] \frac{\exp(ikr)}{4\pi\epsilon_0 r}, \tag{39}$$

where the amplitude of the dipole moment $\mathbf{p}_0$ is proportional to the amplitude of incident wave. When compared to Eq. (32), we obtain

$$f(0)\mathbf{E}_0 = \frac{k^2}{4\pi\epsilon_0} \mathbf{p}_0. \tag{40}$$

The steady-state amplitude of an oscillating electron with a single natural frequency ω_0 and damping γ driven at the frequency ω gives

$$\mathbf{p}_0 = \frac{e^2}{m} (\omega_0^2 - \omega^2 - i\omega\gamma)^{-1} \mathbf{E}_0. \tag{41}$$

Using the last four equations, we obtain the index of refraction[17]

$$n = 1 + \frac{Ne^2}{2\epsilon_0 m(\omega_0^2 - \omega^2 - i\omega\gamma)}, \tag{42}$$

which corresponds to the usual result for a dilute gas of bound electrons. Notice that the small damping term γ, which in general depends on frequency, will cause scattering of the beam out of the forward direction. This term is usually neglected if the frequency is far from resonance.

Although the extinction theorem applies to macroscopic electricity and magnetism in which the medium can be treated as a continuum on the scale of a wavelength, we see that an analogous extinction takes place in a medium which is continuous on a scale much larger than a wavelength. In this calculation, it is clear that the cancellation and replacement of the incident wave by a coherent transmitted wave, traveling in the forward direction with speed c/n, is due to interference of the scattered radiation and the incident wave. For the case of a continuous medium considered earlier, however, there is the cancellation of the coherent superposition of waves traveling in directions other than forward. This cancellation does not apply here.

V. EXTINCTION LENGTH

There has been some disagreement in the literature on the estimation of an extinction length, that is, a distance into the medium for which one can say that the original wave has been replaced by the new wave with index of refraction n. The interest in this question arises from experiments designed to test Einstein's second postulate that the speed of light is independent of the motion of the source. For example, one might observe an object at astronomical distances, but the question arises of whether or not the source of the light received is the object itself or the intervening interstellar medium.

As was stated earlier, the extinction theorem is a theorem about macroscopic electrodynamics in which one averages the fields due to all the constituents of the medium in some volume. Thus, in all cases, there is an averaging length which could be taken as the extinction length. In Sec. II we treated a continuous medium in which one could say that extinction occurs immediately upon the wave entering the medium. But the boundary is sharp only on the scale of the averaging length necessary for the macroscopic Maxwell's equations. On the other hand, the analysis of Sec. IV of the dilute random medium suggests that an appropriate extinction length would be the ''infinitesimal'' length dz which contains a sufficient number of particles so that averaging of the fields makes sense. This length, as stated in the analysis, is much larger than the interparticle distance which is much larger than the wavelength. The two approaches have in common the notion that any extinction length must be large enough for a reasonable average of the electrical properties of the medium. Call this averaging length, extinction length L_1.

Jackson,[18] in his derivation of the macroscopic Maxwell's equations from the microscopic fields, makes an estimate of the size of the minimum averaging length by noting that reflection and refraction of visible light is adequately described by the Maxwell's equations with a continuous dielectric constant but x-ray diffraction shows the atomic nature of matter. He chooses a length of the order of 100 Å in ordinary material for which a cube of that length has the order of 10^6 electrons and nuclei. So it seems reasonable, for both the continuous dielectric and the dilute random medium, to choose an averaging length such that the volume given by the length cubed contains the order of 10^6 particles.

On the basis of such estimates, it seems impossible to check Einstein's second postulate under most circumstances because of the extinction due to any intervening material. But is the length L_1 the correct length to use?

Jackson[12] calculates an ''extinction'' length based on the interpretation that the extinction is caused by a dipole layer on the boundary of the medium.[1,3] He then looks for a distance into the medium where the vacuum wave and the medium wave get significantly out of phase (of order one) and obtains a length L_2,

$$L_2 = \frac{k}{2\pi N|f(0)|} = \frac{\lambda}{2\pi|n-1|}, \tag{43}$$

where the index of refraction is taken real but may be less than one. This result has been criticized because the interpretation that the extinction is caused by a boundary layer is a mathematical artifact due to the changing of a volume integral into a surface integral.

The general view is that the extinction is due to all the oscillating dipoles throughout the volume of the dielectric. Furthermore, this length, L_2, does not appear to be related to the extinction theorem itself since it is so much longer than L_1, the distance needed for averaging the properties of the medium. As will be seen below, however, the length L_2 can be interpreted as the distance in which that portion of the beam that has not interacted with the medium is down by the factor $(1/e)$.

The notion of an ''extinction'' distance suitable for the interpretation of experiments testing the second postulate relies on the picture of the progression of the wave into the medium, interacting with the medium as it goes, until the medium itself must be considered the sole source of the radiation. The extinction theorem of Sec. II, however, deals with stationary waves and so is not suitable for the analysis of this situation. But the formalism for a dilute medium in Sec. IV, based on the idea of repeated scattering of the light as it progresses into the medium, may give us the insight necessary to calculate an appropriate length in which most of the incident radiation has interacted with the medium. Certainly it would not be dz or equivalently L_1, the distance needed for averaging, since, in order to neglect double scattering, it is assumed that only a negligible fraction of the incident wave has been scattered in such a distance.

Suppose the velocity of the radiation in an experimental setup depends on the velocity of the source and is not c. Then the radiation, in passing through a stationary medium, may become extinguished in the process of the forward coherent scattering so that the radiation exiting the medium in the forward direction will be completely scattered radiation. Therefore, measurement of the velocity will yield c since the source of the radiation is the stationary medium. In this case we can estimate a length in which most of the beam has interacted with the medium by following the arguments of Filippas and Fox.[19] The beam traveling through the medium will consist of two components, the component scattered in the forward direction traveling at speed c and the unscattered component (incident wave) at a speed different than c. Note that in Sec. IV, both of these components travel with speed c and it is their interference which gives the refracted wave traveling at speed c/n. Now if the two components of the beam, both in the forward direction, can be differentiated on the basis of speed, then it makes sense to talk about the decrease of the incident beam with distance into the medium. The unscattered fraction of the incident wave can then be measured by itself and its amplitude will decrease exponentially. From Eq. (35) we deduce that the fractional change in the amplitude of the incident wave will be[20]

$$-|n-1|k\,dz \tag{44}$$

in traveling a distance dz through the medium. Here $f(0)$ is written in terms of the index of refraction given in Eq. (38).

This fractional change implies that the scattering results in an exponential decrease, $\exp[-|n-1|kz]$, of the component consisting of the incident wave. This may be used to estimate a distance in which the amplitude of this component has been reduced to $1/e$ of its original value. This e-folding distance is identical to L_2, the ''extinction'' length given by Jackson. It should again be emphasized that the above result depends on detection that separates the unscattered from the scattered wave by their different velocities since both waves move in the forward direction.

Filippas and Fox find the length L_2 (divided by 2 since they calculated the distance in which the intensity rather than the amplitude of the component that has not interacted with the medium is reduced by $1/e$) for 0.5 MeV γ rays to be 19 cm in air and 0.3 mm in lucite. With these results they criticize several experiments purporting to support Einstein's second postulate since, in these experiments, the intervening medium was of the order of three or more e-folding lengths.

Before concluding this section we define another extinction length based on the Rayleigh scattering coefficient for which the total intensity is down by a factor of $1/e$. We assume that any damping of the oscillating dipoles is due to radiative reaction. Then, for a random medium, the intensity of the beam will exponentially decrease with distance as energy is scattered out of the forward direction. The reciprocal of the Rayleigh scattering coefficient[21] will give the distance into the medium for which the intensity of the beam is down by the factor $1/e$. We have

$$L_3 = 6\pi N(c/\omega)^4 \frac{n}{(n^2-1)^2}, \tag{45}$$

valid for frequencies far from resonance, $|\omega-\omega_0| \gg \gamma/2$. In this equation, n is the real part of the index of refraction.

To compare the lengths L_2 and L_3, take the case of a dilute gas of unbound or weakly bound electrons with index of refraction given by Eq. (42). To lowest order in N,

$$L_2 = 1/(\lambda N r_0), \tag{46}$$

where r_0 is $e^2/(4\pi\epsilon_0 mc^2)$, the classical radius of the electron, and

$$L_3 = 1/(N\sigma_T), \tag{47}$$

where σ_T is $8\pi r_0^2/3$, the Thomson cross section. Notice that L_3 is the length of a column of particles for which the sum of the cross sections of the particles equals the cross-sectional area of the column. Therefore, unlike the distance L_2, at the distance L_3 we would expect that all of the radiation has interacted with the medium.

The length L_3, as given in Eq. (47), can be obtained by noting that a complex index of refraction will cause an exponential decay of the beam with distance. This comes from the spatial dependence of the wave as $\exp[inkz]$ where k is the wave number in vacuum. From the index of refraction, Eq. (42), with the radiative damping[22] given by $\gamma = 2r_0\omega^2/3c$, one can easily obtain Eq. (47). The reader may have noticed the peculiarity that this calculation relies on the imaginary part of the index of refraction while the Rayleigh formula depends only on the real part. But the real and imaginary parts are related through causality.[23]

As a numerical example, compare the two lengths for visible light of wavelength 5000 Å. Then, in meters, L_2 is $7.1 \times 10^{12}/N$ while L_3 is $1.5\times 10^{20}/N$, where N is the number of electrons per cubic meter. It is apparent that the distance for the intensity of the beam to decrease to $1/e$ of its original value is much, much, greater than the distance for which all but $1/e$ of the incident radiation has interacted with the medium. There is no appreciable change in the intensity in the distance L_2.

VI. CONCLUSION

We have given an elementary demonstration of the extinction theorem using only the macroscopic fields. This should be more accessible to undergraduates since it relies only on the principle of superposition and the solution of the wave equation. At a more advanced level we have also found a solution in the form of an integral equation which helps to relate our work with earlier studies. We also study the case of short wavelength in a random medium and here it is easy to see how extinction results from the interference of the incident wave and the scattered wave. Finally, we discuss the notion of extinction length which appears to have several different meanings. The extinction length L_1 associated with the theorem is simply the averaging distance necessary to discuss macroscopic fields. But another length, L_2, can be defined as the depth within the medium at which one can say that most of the incident wave has interacted with the medium. Such a question only makes sense if that portion of the wave that has interacted with the medium can be distinguished from the incident wave. But that is the hypothesis of experiments to test the second postulate of Einstein. This length is much larger than the averaging length and is agreement with the ''extinction'' length given by Jackson. For a dilute random medium one can define a third length, L_3, in which the forward intensity of the beam is down by the factor $1/e$ due to scattering out of the forward direction. This length is many orders of magnitude greater than L_2.

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[14] David J. Griffiths, *Introduction to Electrodynamics* (Prentice–Hall, Englewood Cliffs, NJ, 1989) 2nd ed., p. 366.

[15] Bernard Friedman, *Principles and Techniques of Applied Mathematics* (Wiley, New York, 1956), Chap. 3.

[16] Roger G. Newton, *Scattering Theory of Waves and Particles* (Springer-Verlag, New York, 1982), 2nd ed., pp. 23–26. Newton denotes the scattering amplitude by a script A.

[17] Reference 14, pp. 380–385 and 404.

[18] Reference 12, pp. 226–235.

[19] T. A. Filippas and J. G. Fox, ''The velocity of Gamma Rays from a Moving Source,'' Phys. Rev. **135**, B1071 (1964).

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